Spin correlation function in 2D statistical mechanics models with inhomogeneous line defects

Carlos Naón[1] and Marta Trobo[1]

[1]Instituto de Física La Plata, CCT La Plata, CONICET and Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, CC 67, 1900 La Plata, Argentina

(Dated: December, 2010)

We consider the critical spin-spin correlation function of the 2D Ising model with a line defect which strength is an arbitrary function of position. By using path-integral techniques in the continuum description of this model in terms of fermion fields, we obtain an analytical expression for the correlator as functional of the position dependent coupling. Thus, our result provides one of the few analytical examples that allows to illustrate the transit of a magnetic system from scaling to non-scaling behavior in a critical regime. We also show that the non-scaling behavior obtained for the spin correlator along a non-uniformly altered line of an Ising model remains unchanged in the Ashkin-Teller model.

PACS numbers: 05.50.+q, 64.60.De, 75.10.Hk

I. INTRODUCTION

Two dimensional statistical mechanics systems play a central role in our present understanding of phase transitions and critical phenomena. Outstanding members of this family of theories are the Ising model, the Ashkin-Teller[1] and the eight-vertex[2] models. These models are useful to shed light on a variety of phenomena, in both classical and quantum physics, ranging from biological applications[3] to the theory of cuprate superconductors[4]. Moreover, important advances in material science, accomplished over the last decades, have developed the ability to grow and experimentally explore ultrathin ferromagnetic films[5], giving the opportunity to test some of the theoretical predictions. One of the fundamental questions concerning these essentially 2D materials is the role of defects and impurities in the critical properties of magnetic systems. Apart from academic interest, a detailed knowledge on the influence of defects on physical properties is always useful on general grounds, since all real materials are, to some extent, defected. In some cases of applied interest, such as ultrahigh-density magnetic recording media, it has been shown that linear defects can be used to efficiently control domain wall pinning, thus stabilizing the large area domain structure of ultrathin films[6]. Linear charge defects may also appear in graphene grown by chemical vapor deposition on Ni surface[7].

On the theoretical side, very little is known exactly about the behavior of planar systems in the presence of line defects[8]. For the simple square Ising lattice with an altered row (Bariev's model[9]) it has been shown that the scaling index of the magnetization varies continuously with the defect strength[9,10], whereas the critical exponent of the energy density at the defect line remains unchanged[11–13]. Taking this model as working bench, much insight was obtained about the origin of nonuniversal critical behavior[14]. More recently, by using path-integrals within the continuous formulation of Ashkin-Teller and Baxter models, it was shown that the magnetic exponent depends on the strength of the defect in exactly the same way as in Bariev's model[15].

From another perspective, due to the well-known connection between the classical 2D Ising model and a quantum field theory of Dirac fermions in $1+1$ dimensions, the study of defects as perturbations of conformal field theories has led to very important results in the area of integrable quantum field theories[16–18]. This line of research was later focused on the problem of conductance in quantum wires[19]. The analysis of more mathematical aspects concerning the role of impurities and defects in the renormalization group flows of conformal models is currently under intense investigation[20,21]. Very recently, the entanglement between two pieces of a quantum chain was analyzed by exploiting the connection with an Ising model with a defect line[22].

All these advances were achieved for the case of homogeneous defects, i.e. when the defect strength is constant along the altered line. The case of non uniform couplings has been analyzed in the context of extended defects at surfaces[23–25] and in the bulk[26,27], displaying a rich variety of behaviors in the local critical properties.

In this work we consider a narrow inhomogeneous defect and study the spin-spin correlator on the altered line. In other words, we analyze the extension of Bariev's model to the case in which the strength of the line defect is a function of the position on the column with modified couplings. Then, our result for the critical spin-spin correlator is a generalization of the result first obtained in Ref.[10] for a uniform line defect. By using a path-integral approach in the continuum limit, we have obtained a formula that gives the spin-spin correlation function as a functional of an arbitrary defect distribution. This allows to explore the effect of different types of specific alterations in a straightforward way. We have also shown that the results remain valid for the Ashkin-Teller model, i.e. we found

that in these altered systems the non-scaling behavior of magnetic correlations on the inhomogeneous defect coincides with the one obtained in the Ising case.

The paper is organized as follows. In Section II we explain our computational procedure for the well-known defect-free Ising model. In Section III we show how to extend the method when a line of altered couplings is included in the system. We emphasize how the case of inhomogeneous defect strength can be naturally considered with our technique. In Section IV we illustrate the use of our result showing the predictions for two specific defect functions. In Section V we extend the procedure to the more complex Ashkin-Teller and Baxter models. Finally in Section VI we summarize our findings and present our conclusions.

II. THE METHOD: DEFECT-FREE CASE

For completeness and illustrative purposes, we start by describing the computational procedure for the homogeneous defect-free case. The Hamiltonian of the original lattice model is given by

$$\mathcal{H} = -\sum_{<ij>} J_2\, \sigma_i \sigma_j \tag{1}$$

where $< ij >$ means that the sum runs over nearest neighbors of a square lattice ($\sigma = \pm 1$).

As shown in Ref. [28] the scaling regime of the 2D IM can be described in the continuum limit in terms of a model of Majorana fermions with Lagrangian density:

$$\mathcal{L}[\alpha] = \bar{\alpha} i \partial\!\!\!/ \alpha \tag{2}$$

where α represents a Majorana spinor with components $\alpha_{1,2}$. Let us recall that this components are connected to fermion annihilation and creation operators c_r and $c_r^\dagger$ attached to site r ($c_r = \frac{e^{-i\pi/4}}{\sqrt{2}}(\alpha_1(r) + i\alpha_2(r))$. It is also useful for later convenience to define the energy-density as $\epsilon_\alpha = \alpha_1 \alpha_2$. The symbol $\partial\!\!\!/$ stands for $\gamma_\nu \partial_\nu$, with γ_ν the usual Euclidean Dirac matrices ($\nu = 0, 1$ associated to space directions).

Similar manipulations, based on the Jordan-Wigner transformation[29], allow to write the on-line spin-spin correlation function in the form[28]

$$< \sigma(0)\sigma(R) >_{Ising} = < \exp\,(\pi \int_0^R dx\, \epsilon_\alpha(x)) > \tag{3}$$

where the vacuum expectation value is an anticommuting path-integral to be evaluated with the continuum action $S = \int d^2x\, \mathcal{L}$, with an integration measure $\mathcal{D}\alpha$. The explicit computation of (3) can be performed either in terms of the Majorana α-fields or in terms of Dirac fermions[30] built through the doubling technique[31], yielding the well-known result for the Ising correlator. We start by squaring (3):

$$< \sigma(0)\sigma(R) >^2_{Ising} = < \exp\,\left(\pi \int_0^R dx\, (\epsilon_\alpha(x) + \epsilon_{\alpha'}(x))\right) > \tag{4}$$

where the vacuum expectation value must now be computed with respect to an Euclidean action with Lagrangian density $\tilde{\mathcal{L}}[\alpha, \alpha'] = \mathcal{L}[\alpha] + \mathcal{L}[\alpha']$, α' being the replicated fermion fields. Following Ref.[30] we can build Dirac fermions Ψ as

$$\Psi = \alpha + i\alpha'. \tag{5}$$

In terms of these new fields we can write the Lagrangian density $\tilde{\mathcal{L}}[\alpha, \alpha']$ in the form

$$\tilde{\mathcal{L}}[\Psi] \;= \bar{\Psi} i \partial\!\!\!/ \Psi, \tag{6}$$

where $\gamma_5 = i\gamma_0\gamma_1$. On the other hand equation (4) can be expressed as

$$< \sigma(0)\sigma(R) >^2_{Ising} = < \exp\,\left(\pi \int d^2x\, \bar{\Psi}\, A\!\!\!/\, \Psi\right) >, \tag{7}$$

where now the path integral integration measure in the right hand side is expressed in terms of the fields Ψ, and A_ν is an auxiliary vector field with components:

$$A_0(x_0, x_1) = -\delta(x_0)\theta(x_1)\theta(R - x_1),\;\; A_1(x_0, x_1) = 0. \tag{8}$$

Gathering the above results we can write:

$$< \sigma(0)\sigma(R) >^2_{Ising} = \frac{Z[g=\pi]}{Z[g=0]}, \tag{9}$$

where

$$Z[g] = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \, \exp\big(-\int d^2x \big(\tilde{\mathcal{L}}[\Psi] + g\bar{\Psi}\not{A}\,\Psi\big)\big). \tag{10}$$

The continuum limit of the squared two-point spin correlation function is *exactly* expressed in terms of the vacuum to vacuum functional of a quantum field theory describing a Dirac fermion interacting with a classical background A_ν. Now we make the following change of path-integral variables in the numerator of equation (9), with chiral and gauge parameters Φ and η, respectively:

$$\Psi = e^{-\pi(\gamma_5\Phi - i\eta)}\,\zeta, \;\; \bar{\Psi} = \bar{\zeta}\,e^{-\pi(\gamma_5\Phi + i\eta)}. \tag{11}$$

The integration measures $\mathcal{D}\Psi$ and $\mathcal{D}\zeta$ are related through the so called Fujikawa Jacobian J, $\mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi = J[\Phi,\eta]\,\mathcal{D}\bar{\zeta}\,\mathcal{D}\zeta$. If the parameters of the transformation are related to the previously introduced vector field A_ν in the form

$$A_\nu = \epsilon_{\nu\rho}\partial_\rho\Phi + \partial_\nu\eta \tag{12}$$

one easily gets $Z[g=\pi] = J\,Z[g=0]$, which leads to

$$< \sigma(0)\sigma(R) >^2_{Ising} = J(R), \tag{13}$$

As explained in Ref. [32], the Jacobian $J(R)$ must be computed with a gauge-invariant regularization prescription in order to avoid an unphysical linear divergence. Following this procedure one finds that J depends on the Φ-field only as

$$J(R) = \exp -\pi/2 \int d^2x\,\partial_\nu\Phi(x,R)\,\partial^\nu\Phi(x,R). \tag{14}$$

The explicit form of $\Phi(x,R)$ is determined by combining Eqs. (8) and (12) which gives the following partial differential equation for Φ:

$$\Box\Phi(x_0,x_1,R) = -\delta(x_0)\,\frac{d}{dx_1}[\theta(x_1)\theta(R-x_1)] \tag{15}$$

where $\Box = \partial_0^2 + \partial_1^2$. The solution of this equation is easily obtained by using the Green's function of the D'Alembertian: $G_0(z_0,z_1) = \frac{1}{4\pi}\ln(z_0^2 + z_1^2 + a^2)$, with a an ultraviolet cutoff related to the original lattice spacing. Replacing in (14) and considering the limit $R >> a$ we find the well-known result $< \sigma(0)\sigma(R) >_{Ising} \simeq (a/R)^{1/4}$.

III. INHOMOGENEOUS LINE DEFECT

Now we include a line defect in the original Ising lattice. To be specific we consider the so called chain defect (here we employ the terminology of Ref. [8], which corresponds to Bariev's second type defect, in which bonds along the same column are replaced: $J_2 \to J_2'$). In previous studies the altered coupling J_2' was taken as a constant. From now on we allow J_2' to vary from site to site, i.e. we make $J_2' \to J_2'(x_1)$.

We will study the two-spin correlation function in the column of altered bonds $(x_0 = 0)$[10]. It is known that the continuous version of the classical model is modified, due to the defect, by the addition in equation (2) of a term $2\pi\mu(x_1)\,\delta(x_0)\,\epsilon_\alpha(x)$, with $\mu = J_2'(x_1) - J_2$ (see for instance [13]). By carefully examining the fermionic representation of σ-spin operators on the lattice, following the lines of Ref. [30], one also finds that in the continuum limit each spin operator on the defect line picks up a similar μ-dependent factor, in such a way that the correlator for the defective model is given by a simple modification of equation (3):

$$< \sigma(0)\sigma(R) >_{inhom} = < \exp\big(\pi \int dx_1\,(1 + 4\mu(x_1))\,\epsilon_\alpha(x_1)\big) >_\mu \,. \tag{16}$$

It is evident that the squared correlator can be written again as in equation (7). The presence of the inhomogeneous defect manifests in the form of the A_ν-field which is now rescaled by a factor $(1+4\mu(x_1))$. Thus equation (8) becomes

$$A_0(x_0,x_1) = -(1+4\mu(x_1))\delta(x_0)\theta(x_1)\theta(R-x_1) \tag{17}$$

$$A_1(x_0,x_1) = 0. \tag{18}$$

The implementation of the change of variables given by (11) and (12) leads to the generalization of (13):

$$< \sigma(0)\sigma(R) >^2_{inhom} = J_{inhom}(R). \tag{19}$$

Formally $J_{inhom}(R)$ is still given by (14), but the effects coming from the nonuniformity of the defect strength enters the game through the function $\Phi(x_0,x_1,R)$, which now obeys a non trivial differential equation depending on $\mu(x_1)$:

$$\Box\Phi(x_0,x_1,R) = -\delta(x_0)\,\frac{d}{dx_1}[(1+4\mu(x_1))\,\theta(x_1)\,\theta(R-x_1)]. \tag{20}$$

The formal solution of this equation is

$$\begin{aligned}\Phi(x_0,x_1,R) = \tfrac{1}{4\pi}\ln\tfrac{x_0^2+a^2+(x_1-R)^2}{x_0^2+a^2+x_1^2} + \\ +\tfrac{1}{\pi}\int_0^R dx'_1\,\mu(x'_1)\,\tfrac{d}{dx'_1}\ln[x_0^2+(x_1-x'_1)^2+a^2].\end{aligned} \tag{21}$$

Replacing in the corresponding expression for $J_{inhom}(R)$ we obtain

$$< \sigma(0)\sigma(R) >_{inhom} = (\tfrac{a^2}{a^2+R^2})^{\frac{1}{8}+\frac{\mu(0)+\mu(R)}{4}}\,e^{F(R)}, \tag{22}$$

where

$$\begin{aligned}F(R) = \tfrac{1}{4}\int_0^R dx\,\mu(x)\,\tfrac{d}{dx}\big[\ln\tfrac{(a^2+(x-R)^2)^{(1+4\mu(R))}}{(a^2+x^2)^{(1+4\mu(0))}}\big] - \\ -\tfrac{1}{4}\int_0^R\int_0^R dx\,dy\,(1+4\mu(x))\,\tfrac{d}{dy}\mu(y)\,\tfrac{d}{dx}\big[\ln(a^2+(x-y)^2)\big].\end{aligned} \tag{23}$$

In the above integrals we have dropped the subindex 1 in the integration variables, in order to simplify the notation ($x_1 \to x$ and $y_1 \to y$). It is easy to check that in the special case $\mu(x) \to \mu = constant$, one obtains

$$< \sigma(0)\sigma(R) >_\mu \simeq (\frac{a}{R})^{2\Delta_\sigma}, \tag{24}$$

with $\Delta_\sigma = \frac{1}{8}(1+4\mu)^2$, which is the well-known result first obtained by McCoy and Perk[10].

Formulae (22) and (23) constitute the main formal result of this paper. They give the critical spin-spin correlation on the altered line of an Ising model, as a functional of an arbitrarily varying defect strength. In the next section we will show some specific predictions for definite defect distributions.

IV. APPLICATION TO SOME SPECIFIC DEFECTS

Let us now consider some specific defect-functions for which $F(R)$ can be analytically evaluated. We start with the following defect distribution

$$\mu(x) = \mu_0\frac{1}{(1+\mid x\mid/b)}. \tag{25}$$

where b is a characteristic length scale. This function is similar to the one considered by Bariev in his study of horizontal large scale inhomogeneities[26]. Passing to dimensionless variables $r = R/a$ and $\beta = b/a$, and considering weak defect strengths ($\mu_0 \ll 1$), for $R \gg a$ and $b \gg a$ we obtain

$$< \sigma(0)\sigma(r) >_{inhom}= \left(\frac{1}{r}\right)^{\frac{1}{4}+\frac{\mu_0(2\beta+r)}{(\beta+r)}} \left(\frac{\beta}{\beta+r}\right)^{\frac{-\mu_0 r}{(\beta+r)}} \exp^{\left(\frac{\mu_0 r(2\beta+r)}{\beta(\beta+r)^2}\arctan(r)\right)} . \quad (26)$$

We see that the magnetic correlation exhibits non-scaling behavior, as expected for a local inhomogeneity. This result is in qualitative agreement with the analysis of Ref.[26]. However we should stress that we are considering a different situation here. Indeed, the present case corresponds to a standard 2D Ising model in which just one column ($x_0 = 0$) is altered in a non uniform way, whereas in Ref.[26] the couplings along columns are kept constant, while the couplings along *all* rows are modified in a non uniform fashion. In Figure 1 we compare the decays of correlations for constant defect (solid line), non constant defect with decay law (25) (pointed line) and the universal defect-free behavior (dashed line). In agreement with physical intuition the correlation decays monotonically with distance, in an intermediate way, faster than the defect-free case and slower than the case in which the defect strength is constant. Another expected feature, well reproduced by our solution, concerns the behavior with $\beta = b/a$: for increasing β the non-scaling decay becomes faster, being undistinguishable from the uniform case for large enough β.

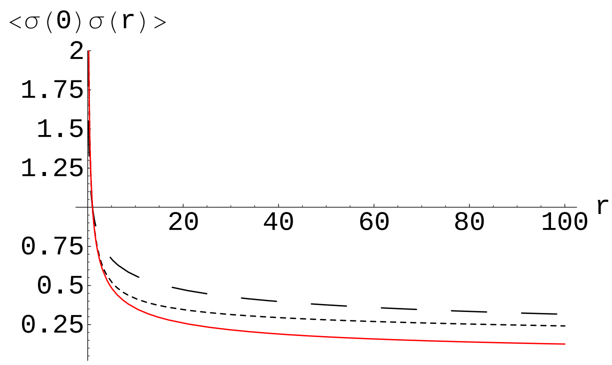

FIG. 1: Spin-spin correlation as a function of r for constant (solid line) and variable (pointed line) defect strength. We set $\beta = 10$. The dashed line indicates the defect-free universal behavior $r^{-1/4}$.

Let us now study a different function $\mu(x)$, which represents a non-monotonic alteration of the line $x_0 = 0$. For simplicity we consider a sequence of N slabs of heights μ_{0i} ($i = 1, ...N$). Each slab starts at $x = ac_i$ and ends at $x = ad_i$:

$$\mu(x) = \sum_{i=1}^{N} \mu_{0i}\, \theta(x - ac_i)\, \theta(ad_i - x), \quad (27)$$

where $\theta(x)$ is Heaviside's function. Evaluating $F(R)$ and replacing in (22), in the weak coupling regime ($\mu_{0i} \ll 1$) and for $r, c_i, d_i \gg 1$ we obtain

$$< \sigma(0)\sigma(r) >_{inhom}= \left(\frac{1}{r}\right)^{1/4} \prod_{i=1}^{N} \left[\left(\frac{1}{r}\right)^2 \left(\frac{c_i^2}{(c_i - r)^2 + 1}\right)\right]^{\mu_{0i}\theta(d_i-r)\theta(r-c_i)/2} \left[\left(\frac{c_i}{d_i}\right)^2 \left(\frac{(d_i - r)^2 + 1}{(c_i - r)^2 + 1}\right)\right]^{\mu_{0i}\theta(r-d_i)/2} \quad (28)$$

In Figure 2 we display the result given by the above formula for the simplest case: one slab or "barrier" starting at $x/a = c = 10$ and ending at $x/a = d = 50$. For $r < c$ the critical two-spin correlation coincides with the standard, non defected correlation. In the presence of the defect, for $c < r < d$, it exhibits a faster decay. The correlation reaches a local minimum at $r = d$, and then it starts growing, approaching again the universal behavior corresponding to the magnetic critical index 1/8, asymptotically. In Figure 3, taking into account that (28) is valid for both positive and negative values of μ_0, we show the critical correlation for a defect which is oscillatory along certain portion of the

line $x_0 = 0$, a sequence of five alternated slabs ($\mu_0 = 0.1$) and wells ($\mu_0 = -0.1$). As before, the spin-spin function coincides with the non defected one, for small distances ($r < c_1$). For $c_1 < r < d_5$ there is an oscillatory behavior around the universal curve $r^{-1/4}$. For large distances the correlation tends to the universal decay.

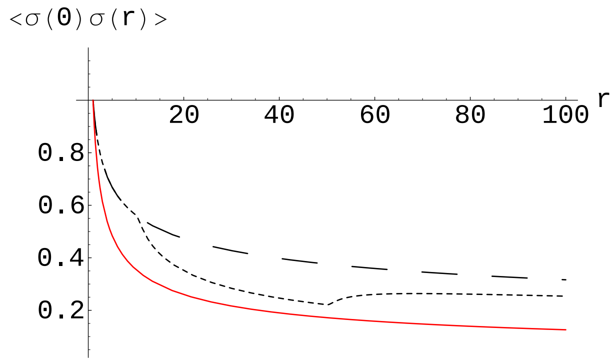

FIG. 2: Spin-spin correlation as a function of r for a line defect given by a slab starting at $x/a = 10$ and ending at $x/a = 50$, for $\mu_0 = 0.1$ (pointed line). The dashed line indicates the defect-free universal behavior $r^{-1/4}$. The solid line corresponds to a uniform defect with $\mu_0 = 0.1$.

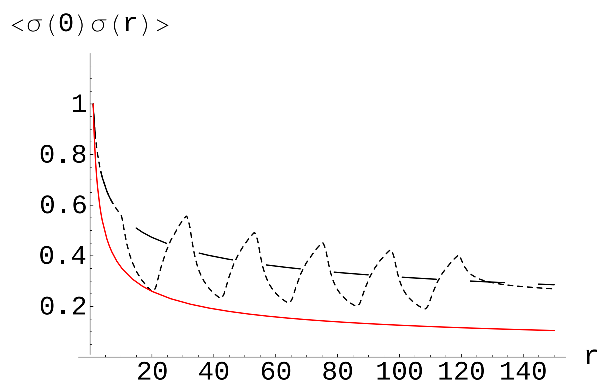

FIG. 3: Spin-spin correlation as a function of r for a line defect given by an oscillatory function (a sequence of 5 slabs and wells with equal heights (depths) and widths ($\mid \mu_0 \mid= 0.1$),starting at $x/a = 10$ and ending at $x/a = 119$) (pointed line). The dashed line indicates the defect-free universal behavior $r^{-1/4}$. The solid line corresponds to a uniform defect with $\mu_0 = 0.1$.

V. EXTENSION TO THE ASHKIN-TELLER MODEL

In this Section we show how to extend the treatment of inhomogeneous linear impurities to the Ashkin-Teller system consisting of two Ising subsystems with spin variables σ_i and τ_i coupled by a quartic interaction[12]. The corresponding lattice Hamiltonian reads

$$\mathcal{H} = -\sum_{<ij>} \left[J_2 \left(\sigma_i \sigma_j + \tau_i \tau_j \right) + J_4 \, \sigma_i \sigma_j \tau_i \tau_j \right] \tag{29}$$

where $< ij >$ means that the sum runs over nearest neighbors of a square lattice ($\sigma, \tau, = \pm 1$). As it is well known, in the vicinity of the critical point this model can be described in the continuum limit in terms of two Majorana fermions interacting via their energy-densities:

$$\mathcal{L}[\alpha, \beta] = \bar{\alpha} i \partial\!\!\!/ \alpha + \bar{\beta} i \partial\!\!\!/ \beta - \lambda \, \epsilon_\alpha \, \epsilon_\beta \tag{30}$$

where α and β are the Majorana spinors with components $\alpha_{1,2}$, $\beta_{1,2}$ respectively. $\epsilon_\alpha = \alpha_1\alpha_2$ and $\epsilon_\beta = \beta_1\beta_2$ are the corresponding energy-densities. The coupling constant λ is proportional to J_4/J_2. Let us now include, as before, a linear defect affecting one of the original Ising lattices, say the one with spins σ. If this impurity is placed at column $x_0 = 0$, in the continuum limit we have to add to $\mathcal{L}$ a term $2\pi\mu(x_1)\,\delta(x_0)\,\epsilon_\alpha(x)$, with $\mu = J_2' - J_2$. As shown in Ref. 15, in order to compute the spin-spin correlator on the altered line is still possible to use the doubling technique depicted in Section II. However, in spite of the formal analogy, the situation is much more complex here. First of all, since we have two sets of spins, we have to introduce two Dirac fields: $\Psi = \alpha + i\alpha'$ and $\chi = \beta + i\beta'$. We then obtain

$$< \sigma(0)\sigma(R) >_{AT}^2 = < \exp\left(\pi \int d^2x \; \bar{\Psi} \not{A}\, \Psi\right) >_\mu . \tag{31}$$

Here the background field A_ν is given by (17) and the vacuum expectation value must be computed with respect to an Euclidean action with Lagrangian density $\mathcal{L}[\Psi,\chi]$:

$$\begin{aligned} \tilde{\mathcal{L}}[\Psi,\chi] &= \bar{\Psi} i\not{\partial}\Psi + \bar{\chi} i\not{\partial}\chi \\ &- \frac{\lambda}{8}[\bar{\chi}\gamma_5\chi\bar{\Psi}\gamma_5\Psi + Im(\chi^T\gamma_1\chi)Im(\Psi^T\gamma_1\Psi)], \end{aligned} \tag{32}$$

where $\gamma_5 = i\gamma_0\gamma_1$ and Ψ^T, χ^T are the transposed spinors.

The implementation of the change of variables given by (11) and (12) leads to

$$< \sigma(0)\sigma(R) >_{AT}^2 = < \sigma(0)\sigma(R) >_{inhom}^2 F(\lambda, R, \mu) \tag{33}$$

where $< \sigma(0)\sigma(R) >_{inhom}$ is the defected Ising correlator given in (22) and

$$F(\lambda, R, \mu) = \mathcal{N}(\lambda) < \exp[S_\Phi(\zeta,\chi) + S_\eta(\zeta,\chi)] >_0 \tag{34}$$

where $<>_0$ means vacuum expectation value with respect to the model of free χ and ζ fermions. $\mathcal{N}(\lambda)$ is a normalization constant independent of R. Since the analysis of the dependence of $F(\lambda,\mu,R)$ on R is more easily done in momentum space, we have Fourier transformed $S_\Phi(\zeta,\chi,\mu)$ and $S_\eta(\zeta,\chi,\mu)$ in the above equation:

$$S_\Phi(\zeta,\chi,\mu) = \frac{\lambda}{8}\int \prod_{j=1}^{4} \frac{d^2p_j}{(2\pi)^2}[\bar{\chi}(p_1)\gamma_5\chi(p_2)\bar{\zeta}(p_3)\gamma_5 G(P,R,\mu)\zeta(p_4), \tag{35}$$

with $G(P,R,\mu)$ being a diagonal 2x2 matrix given by

$$G(P,R,\mu) = \begin{pmatrix} g_+(P,R,\mu) & 0 \\ 0 & g_-(P,R,\mu) \end{pmatrix}, \tag{36}$$

where $g_\pm(P,R,\mu) = \pm\int d^2x e^{iPx} e^{\mp 2\pi\Phi(x,\mu,R)}$ and $P = p_1 + p_2 + p_3 + p_4$. A similar expression is obtained for S_η with $G(P,R)$ replaced by

$$H(P,R,\mu) = \begin{pmatrix} h(P,R,\mu) & 0 \\ 0 & h(P,R,\mu) \end{pmatrix}, \tag{37}$$

with $h(P,R,\mu) = \int d^2x e^{iPx} e^{2\pi\eta(x,R,\mu)}$. The explicit functional forms of $\Phi(x,R,\mu)$ and $\eta(x,R,\mu)$ can be determined following the same steps depicted in previous Sections, yielding

$$\begin{aligned} \Phi(x_0,x_1,R,\mu) &= \tfrac{-1}{4\pi}\ln\left(\tfrac{x_0^2+a^2+(x_1-R)^2}{x_0^2+a^2+x_1^2}\right) + \\ &+ \tfrac{2}{\pi}\int_0^R dx_1'\,\mu(x_1')\,\tfrac{(x_1-x_1')}{(x_0^2+(x_1-x_1')^2+a^2)} \end{aligned} \tag{38}$$

and

$$\eta(x_0,x_1,R,\mu) = \tfrac{x_0}{2\pi}\int_0^R dy \tfrac{(1+4\mu(y))}{(x_0^2+a^2+(y-x_1)^2)}. \tag{39}$$

Then, $g(P,R,\mu)$ becomes

$$g_{\pm}(P,R,\mu) = \pm\int d^2x e^{iPx}\Big(\frac{x_0^2+a^2+(x_1-R)^2}{x_0^2+a^2+x_1^2}\Big)^{\pm 1/2} e^{\mp 4\int_0^R dy\mu(y)\frac{(x_1-y)}{(x_0^2+a^2+(y-x_1))^2}} \tag{40}$$

and $h(P,R)$

$$h(P,R,\mu) = \int d^2x e^{iPx} e^{ix_0/\sqrt{x_0^2+a^2}\arctan\left(\frac{R\sqrt{x_0^2+a^2}}{x_0^2+x_1^2+a^2-Rx_1}\right)} e^{i4x_0\int_0^R dy\frac{\mu(y)}{(x_0^2+a^2+(y-x_1)^2)}}. \tag{41}$$

Since any possible dependence on R of the function $F(\lambda,R,\mu)$ comes from $g_{\pm}(P,R,\mu)$ and $h(P,R,\mu)$, our problem is reduced to the analysis of these integrals. Let us first introduce a cutoff L, which can be interpreted as the size of the system, in order to avoid infrared divergencies (the thermodynamic limit will be recovered at the end of the computation by setting $L\to\infty$). In terms of the dimensionless variable $u_\rho = x_\rho/L$, $(\rho = 0,1)$ we obtain

$$\begin{aligned} g_{\pm}(P,R,\mu) &= \lim_{L\to\infty} \pm L^2\int_{|u_\rho|<1} d^2u \\ &\times\ e^{iPLu}\Big(\frac{u_0^2+a^2/L^2+(u_1-(R/L))^2}{u_0^2+a^2/L^2+u_1^2}\Big)^{\pm 1/2} e^{\mp\frac{4}{L}\int_0^R dy\mu(y)\frac{(u_1-y/L)}{(u_0^2+a^2/L^2+(y/L-u_1))^2}} \\ &= \pm(2\pi)^2\delta^2(P) \end{aligned} \tag{42}$$

and a similar result for $h(P,R)$. Then, in the thermodynamic limit $(a \ll R \ll L)$ $F(\lambda,R,\mu)$ becomes independent of R and the critical behavior coincides with the one of the 2D Ising model in presence of an arbitrary inhomogeneous defect.

VI. SUMMARY AND CONCLUSIONS

We have considered the critical behavior of the two-spin correlation function in the continuum, field-theory version of the 2D Ising model with a line defect placed at the column $x_0 = 0$. In contrast to previous studies, here we have taken into account possible variations of the defect strength with the position on the line. Our main result (Eqs. (22) and (23)) provides an analytical expression for the critical spin-spin correlation as a functional of an arbitrary defect distribution. From this one can explore the effect of different types of non uniform impurity distributions on the magnetization. In particular our finding can be used to analyze, within the critical regime, the transit from scaling to non scaling behavior. As examples, in order to illustrate the approach and check its validity, we have discussed two special cases: a defect strength decaying monotonously with distance from a given point, and a sequence of slabs. Finally, we extended the analysis to a nonhomogeneous line defect placed at one column of an Ashkin-Teller system, showing that the spin correlator on the altered line decays, in the thermodynamic limit, in the same way as in the Ising model.

Acknowledgement
The authors are grateful to CONICET and UNLP (Argentina) for financial support.

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